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On the Rate of Convergence of the Durrmeyer Operator for Functions of Bounded Variation

SHUNSHENG GUO

Department of Mathematics, Hebei Normal University, Shijiazhuang, Hebei, People's Republic of China

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1. INTRODUCTION

Bojanic estimated in [1] the rate of convergence of Fourier series of functions of bounded variation. Fuhua Cheng gave a result of this type for the Bernstein operator in [2]. We improved the result of [2] and obtained a result of this type for the Kantorovitch operator in [3].

If f is a function defined on [0, 1], the Durrmeyer operator M_n applied to f is

$$M_n(f, x) = (n+1) \sum_{k=0}^{n} p_{nk}(x) \int_0^1 f(t)\, p_{nk}(t)\, dt \tag{1.1}$$

where

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

(see [4]).

In this paper, we shall give an estimate for the rate of convergence of (1.1) for functions of bounded variation. Using some results of probability theory, we shall prove the following:

THEOREM. *Let f be a function of bounded variation on [0, 1] and let $V_a^b(g_x)$ be the total variation of g_x on $[a, b]$. Then, for every $x \in (0, 1)$ and n sufficiently large, we have*

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$$|M_n(f, x) - \tfrac{1}{2}(f(x+) + f(x-))|$$
$$\leqslant \frac{5(x(1-x))^{-1}}{\eta} \sum_{k=1}^{\eta} \bigvee_{x - x/\sqrt{k}}^{x+(1-x)/\sqrt{k}} (g_x)$$
$$+ \frac{13(x(1-x))^{-1}}{4\sqrt{\eta}} |f(x+) - f(x-)| \tag{1.2}$$

where

$$g_x(t) = \begin{cases} f(t) - f(x+) & x < t \leqslant 1 \\ 0 & t = x \\ f(t) - f(x-) & 0 \leqslant t < x. \end{cases}$$

In the last part of this paper, we shall prove that our estimate is essentially the best possible.

2. Lemmas

The proof of the theorem is based on the following lemmas.

LEMMA 1. *If* $\{\xi_k\}$ $(k \geqslant 1)$ *are independent random variables with the same distribution functions and* $0 < D\xi_k < \infty$, $\beta_3 = E(\xi_r - E\xi_i)^3 < \infty$, *then*

$$\max_y \left| P\left(\frac{1}{b_1\sqrt{n}} \sum_{k=1}^{n} (\xi_k - a_1) \leqslant y\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt \right| < \frac{c}{\sqrt{n}} \frac{\beta_3}{b_1^3} \tag{2.1}$$

where $a_1 = E(\xi_1)$ *(expectation of* ξ_1*),* $b_1^2 = D\xi_1 = E(\xi_1 - E\xi_1)^2$ *(variance of* ξ_1*) and* $1/\sqrt{2\pi} \leqslant c < 0.82$ *(see* [5], p. 159 *or* [6], p. 300).

LEMMA 2. *For every* $x \in (0, 1)$, $0 \leqslant k \leqslant n$, *we have*

$$p_{nk}(x) \leqslant \frac{5}{2\sqrt{nx(1-x)}}. \tag{2.2}$$

Proof. Let $\{\xi_k\}$ be a sequence of Bernoulli trials with parameter x; $P(\xi_k = 1) = x \in (0, 1)$, $P(\xi_k = 0) = 1 - x$. Obviously, $a_1 = E\xi_1 = x$, $b_1 = (D\xi_1)^{1/2} = \sqrt{x(1-x)}$, $\beta_3 = x(1-x)(x^2 + (1-x)^2) \leqslant x(1-x)$. Hence

$$\frac{c}{\sqrt{n}} \frac{\beta_3}{b_1^3} \leqslant \frac{1}{\sqrt{nx(1-x)}}. \tag{2.3}$$

Define $\eta_n = \sum_{i=1}^{n} \xi_i$; then η_n is the number of successes in the first n trials and

$$P(\eta_n = k) = \binom{n}{k} x^k (1-x)^{n-k} = p_{nk}(x).$$

On the other hand,

$$p_{nk}(x) = P(k-1 < \eta_n \leqslant k) = P\left(\frac{k-1-nx}{\sqrt{nx(1-x)}} < \frac{\eta_n - nx}{\sqrt{nx(1-x)}} \leqslant \frac{k-nx}{\sqrt{nx(1-x)}}\right).$$

Using Lemma 1 and (2.3), we have

$$\left| p_{nk}(x) - \frac{1}{\sqrt{2\pi}} \int_{k-1-nx/\sqrt{nx(1-x)}}^{k-nx/\sqrt{nx(1-x)}} e^{-t^2/2}\, dt \right| < \frac{2c}{\sqrt{n}} \frac{\beta_3}{b_1^3} < \frac{2}{\sqrt{nx(1-x)}}.$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_{k-1-nx/\sqrt{nx(1-x)}}^{k-nx/\sqrt{nx(1-x)}} e^{-t^2/2}\, dt \leqslant \frac{1}{\sqrt{2\pi nx(1-x)}},$$

(2.2) is proved.

LEMMA 3. *For every $0 \leqslant j \leqslant n$ and n sufficiently large, we have*

$$\left| \sum_{k=0}^{j} p_{nk}(x) - \sum_{k=0}^{j} p_{n+1k}(x) \right| \leqslant \frac{2}{\sqrt{nx(1-x)}}. \tag{2.4}$$

Proof. Since $\sum_{k=0}^{j} p_{nk}(x) = P(\eta_n \leqslant j)$, using Lemma 1, we have

$$\left| \sum_{k=0}^{j} p_{nk}(x) - \frac{1}{\sqrt{2\pi}} \int_{\infty}^{j-nx/\sqrt{nx(1-x)}} e^{-t^2/2}\, dt \right| < \frac{1}{\sqrt{nx(1-x)}}$$

and

$$\left| \sum_{k=0}^{j} p_{n+1k}(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{j-(n+1)x/\sqrt{(n+1)x(1-x)}} e^{-t^2/2}\, dt \right| < \frac{1}{\sqrt{(n+1)x(1-x)}}.$$

Hence

$$\begin{aligned}
&\left| \sum_{k=0}^{j} p_{nk}(x) - \sum_{k=0}^{n} p_{n+1k}(x) \right| \\
&\quad \leqslant \left| \frac{1}{\sqrt{2\pi}} \int_{j-(n+1)x/\sqrt{(n+1)x(1-x)}}^{j-nx/\sqrt{nx(1-x)}} e^{-t^2/2}\, dt \right| + \frac{2}{\sqrt{nx(1-x)}} \\
&\quad \leqslant \frac{1}{\sqrt{2\pi x(1-x)}} \frac{3}{2\sqrt{\pi}} + \frac{1}{\sqrt{nx(1-x)}} \leqslant \frac{2}{\sqrt{nx(1-x)}}.
\end{aligned}$$

LEMMA 4. *For every* $0 \leqslant j \leqslant n$, *we have*

$$\sum_{k=0}^{j} p_{n+1k}(x) = (n+1) \int_x^1 p_{nj}(t)\, dt \tag{2.5}$$

Proof. (2.5) can easily be proved by differentiating both its left-hand and rights-hand sides.

LEMMA 5. *If n is sufficiently large, then*

$$\frac{x(1-x)}{n} \leqslant M_n((t-x)^2, x) \leqslant \frac{2x(1-x)}{n}. \tag{2.6}$$

Proof. By an easy calculation we have

$$M_n(x, x) = x, \qquad M_n(t, x) = \frac{nx+1}{n+2},$$

$$M_n(t^2, x) = \frac{n^2x^2 + nx(1-x) + 3nx + 2}{(n+2)(n+3)}.$$

Hence

$$M_n((t-x)^2, x) = \frac{2[(n-3)\, x(1-x) + 1]}{(n+2)(n+3)}.$$

(2.6) follows at once for n sufficiently large.

LEMMA 6. *Let* $K_n(x, t) = (n+1) \sum_{k=0}^{n} p_{nk}(x)\, p_{nk}(t)$. *If n is sufficiently large, then*

(1) *For* $0 \leqslant y < x$, *we have*

$$\int_0^y K_n(x, t)\, dt \leqslant \frac{2x(1-x)}{n(x-y)^2}. \tag{2.7}$$

(2) *For* $x < z \leqslant 1$, *we have*

$$\int_z^1 K_n(x, t)\, dt \leqslant \frac{2x(1-x)}{n(z-x)^2}. \tag{2.8}$$

Proof. Since $0 \leqslant y < x$, for $t \in [0, y]$ we have

$$\frac{x-t}{x-y} \geqslant 1.$$

By (2.6), for n sufficiently large, we have

$$\int_0^y K_n(x,t)\,dt \leqslant \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_n(x,t)\,dt$$
$$\leqslant \frac{1}{(x-y)^2}\int_0^1 (x-t)^2 K_n(x,t)\,dt$$
$$=\frac{1}{(x-y)^2} M_n((t-x)^2, x) \leqslant \frac{2x(1-x)}{n(x-y)^2},$$

proving (2.7). The proof of (2.8) is similar.

3. Proof of the Theorem

First,

$$|M_n(f,x)-\tfrac{1}{2}(f(x+)+f(x-))|$$
$$\leqslant |M_n(g_x,x)| + \tfrac{1}{2}|f(x+)-f(x-)|\,|M_n(\mathrm{Sign}(t-x),x)|. \quad (3.1)$$

Thus to estimate $|M_n(f,x)-\frac{1}{2}(f(x+)+f(x-)|$ we need estimates for $M_n(g_x,x)$ and $Mn(\mathrm{Sign}(t-x),x)$.

To estimate $M_n(\mathrm{Sign}(t-x),x)$, we first decompose it into two parts as follows:

$$M_n(\mathrm{Sign}(t-x), x = \int_0^1 \mathrm{Sign}(t-x)\,K_n(x,t)\,dt$$
$$= \int_x^1 K_n(x,t)\,dt - \int_0^x K_n(x,t)\,dt \stackrel{\text{def}}{=} A_n(x)-B_n(x).$$

Using Lemma 4, we have

$$A_n(x)=\int_x^1 K_n(x,t)\,dt=(n+1)\sum_{k=0}^n p_{nk}(x)\int_x^1 p_{nk}(t)\,dt$$
$$=\sum_{k=0}^n \left(p_{nk}(x)\sum_{i=0}^k p_{n+1i}(x)\right).$$

By Lemma 3, it follows that

$$\left|A_n(x)-\sum_{k=0}^n\left(p_{nk}(x)\sum_{i=0}^k p_{ni}(x)\right)\right| \leqslant \frac{2}{\sqrt{nx(1-x)}}. \quad (3.2)$$

Let

$$S=\sum_{k=0}^{n}\left(p_{nk}(x)\sum_{i=0}^{k}p_{ni}(x)\right)$$
$$=p_{n0}p_{n0}+p_{n1}(p_{n0}+p_{n1})+\cdots+p_{nn}(p_{n0}+p_{n1}+\cdots+p_{nn}).$$

Since

$$I=(p_{n0}+p_{n1}+\cdots+p_{nn})(p_{n0}+p_{n1}+\cdots+p_{nn})$$
$$=p_{n0}(p_{n0}+p_{n1}+\cdots+p_{nn})+\cdots+p_{nn}(p_{n0}+p_{n1}+\cdots+p_{nn}),$$

we have

$$I-S=p_{n0}(p_{n1}+\cdots+p_{nn})+p_{n1}(p_{n2}+\cdots+p_{nn})+\cdots+p_{nn-1}p_{nn}$$
$$=p_{nn}(p_{n0}+\cdots+p_{nn-1})+p_{nn-1}(p_{n0}+\cdots+p_{nn-2})+\cdots+p_{n1}p_{n0}$$

and

$$2S-I=p_{n0}^2+p_{n1}^2+\cdots+p_{nn}^2.$$

Using Lemma 2, we obtain

$$|S-\tfrac{1}{2}|\leqslant\frac{5}{4\sqrt{nx(1-x)}}\sum_{k=0}^{n}p_{nk}(x)=\frac{5}{4\sqrt{nx(1-x)}}. \tag{3.3}$$

By (3.2) and (3.3) it follows that

$$|A_n(x)-\tfrac{1}{2}|\leqslant\frac{13}{4\sqrt{nx(1-x)}}. \tag{3.4}$$

On the other hand,

$$A_n(x)+B_n(x)=\int_0^1 K_n(x,t)\,dt=1.$$

Hence

$$|A_n(x)-B_n(x)|=|2A_n(x)-1|\leqslant\frac{13}{2\sqrt{nx(1-x)}}. \tag{3.5}$$

The estimate of $M_n(g_x, x)$ is similar to [2]. We first decompose [0, 1] into three parts, as follows:

$$I_1=\left[0, x-\frac{x}{\sqrt{n}}\right],\qquad I_2=\left[x-\frac{x}{\sqrt{n}}, x+\frac{1-x}{\sqrt{n}}\right],\qquad I_3=\left[x+\frac{1+x}{\sqrt{n}}, 1\right].$$

Then

$$M_n(g_x, x) = \int_0^1 g_x(t)\, K_n(x, t)\, dt$$
$$= \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) g_x(t)\, K_n(x, t)\, dt$$
$$\overset{\text{def}}{=} \Delta_{1,n}(f, x) + \Delta_{2,n}(f, x) + \Delta_{3,n}(f, x).$$

Let $\lambda_n(x, t) = \int_0^t K_n(x, u)\, du$.

First, we estimate $\Delta_{2,n}(f, x)$. For $t \in I_2$, we have

$$| g_x(t)| = | g_x(t) - g_x(x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x)$$

and so

$$|\Delta_{2,n}(f, x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x) \int_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}} d_t \lambda_n(x, t).$$

Since $\int_a^b d_t \lambda_n(x, t) \leqslant 1$ for all $[a, b] \subseteq [0, 1]$, therefore

$$|\Delta_{2,n}(f, x)| \leqslant V_{x - x/\sqrt{n}}^{x + (1-x)/\sqrt{n}}(g_x) \tag{3.6}$$

To estimate $\Delta_{1,n}(f, x)$, let $y = x - (x/\sqrt{n})$ and note that g_x is normalized on $(0, 1)$. Using Lebesgue–Stieltjes integration by parts, we find that

$$\Delta_{1,n}(f, x) = \int_0^y g_x(t)\, d_t \lambda_n(x, t)$$
$$= g_x(y+)\, \lambda_n(x, y) - \int_0^y \lambda_n(x, t)\, d_t\, g_x(t).$$

Since

$$| g_x(y+)| = | g_x(y+) - g_x(x)| \leqslant V_{y+}^x(g_x),$$

it follows that

$$|\Delta_{1,n}(f, x)| \leqslant V_{y+}^{+}(g_x)\, \lambda_n(x, y) + \int_0^y \lambda_n(x, t)\, d_t(-V_t^x(g_x)).$$

By (2.7) of Lemma 6, we have

$$|\Delta_{1,n}(f, x)| \leqslant V_{y+}^x(g_x) \frac{2x(1-x)}{n(x-y)^2} + \frac{2x(1-x)}{n} \times \int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)).$$

Furthermore, since

$$\int_0^y \frac{1}{(x-t)^2}\, d_t(-V_t^x(g_x)) = -\frac{V_{y+}^x(g_x)}{(x-y)^2} + \frac{V_0^x(g_x)}{x^2}$$
$$+2\int_0^y \hat{V}_t^x(g_x)\frac{dt}{(x-t)^3}$$

where $\hat{V}_t^x(g_x)$ is the normalized form of $V_t^x(g_x)$ and $\hat{V}_t^x(g_x) = V_t^x(g_x)$, we have

$$|\Delta_{1,n}(f, x)| \leqslant \frac{2x(1-x)}{n}\left(\frac{V_0^x(g_x)}{x^2} + 2\int_0^{x-x/\sqrt{n}} V_t^x(g_x)\frac{dt}{(x-t)^3}\right).$$

Replacing the variable t in the last integral by $x - x/\sqrt{t}$, we find that

$$\int_0^{x-x/\sqrt{n}} V_t^x(g_x)\frac{dt}{(x-t)^3} = \frac{1}{2x^2}\int_1^n V_{x-x/\sqrt{t}}^x(g_x)\, dt$$
$$\leqslant \frac{1}{2x^2}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x).$$

Hence

$$|\Delta_{1,n}(f, x)| \leqslant \frac{2(1-x)}{nx}\left(V_0^x(g_x) + \sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x)\right)$$
$$\leqslant \frac{4(1-x)}{nx}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x)$$
$$\leqslant \frac{4}{nx(1-x)}\sum_{k=1}^n V_{x-x/\sqrt{k}}^x(g_x). \tag{3.7}$$

Using a similar method and (2.8) of Lemma 6, we obtain

$$|\Delta_{3,n}(f, x)| \leqslant \frac{4}{nx(1-x)}\sum_{k=1}^n V_x^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.8}$$

From (3.6), (3.7), and (3.8), it follows that

$$|M_n(g_x, x)| \leqslant \frac{4}{nx(1-x)}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x) + V_{x-x/\sqrt{n}}^{x+(1-x)/\sqrt{n}}(g_x)$$
$$\leqslant \frac{5}{nx(1-x)}\sum_{k=1}^n V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(g_x). \tag{3.9}$$

Our theorem now follows from (3.1), (3.5), and (3.9).

4. Remark

We shall prove that our estimate is essentially the best possible.

Consider the function $f(t)=|t-x|$ $(0<x<1)$ on $[0,1]$. By (2.6), for any small $\delta>0$ and n sufficiently large, we have

$$\begin{aligned} M_n(|t-x|,x) &= (n+1)\sum_{k=0}^{n} p_{nk}(x) \\ &\quad\times\left(\int_{x-\delta}^{x+\delta}+\int_{|t-x|>\delta}|t-x|\,p_{nk}(t)\,dt\right) \\ &\leqslant (n+1)\sum_{k=0}^{n} p_{nk}(x)\left(\frac{\delta}{n+1}+\frac{1}{\delta}\int_0^1 (t-x)^2\,p_{nk}(t)\,dt\right) \\ &\leqslant \delta+\frac{2x(1-x)}{n\,\delta} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} M_n(|t-x|,x) &\geqslant (n+1)\sum_{k=0}^{n} p_{nk}(x)\int_{x-\delta}^{x+\delta}|t-x|\,p_{nk}(t)\,dt \\ &\geqslant \frac{1}{\delta}\left((n+1)\sum_{k=0}^{n} p_{nk}(x)\int_{x-\delta}^{x+\delta}(t-x)^2\,p_{nk}(t)\,dt\right) \\ &\geqslant \frac{x(1-x)}{n\,\delta}-\frac{1}{\delta}\Bigg[(n+1)\sum_{k=0}^{n} p_{nk}(x) \\ &\quad\times\int_{|t-x|>\delta}(t-x)^2\,p_{nk}(t)\,dt \end{aligned}$$

Since

$$\begin{aligned} &(n+1)\sum_{k=0}^{n} p_{nk}(x)\int_{|t-x|>\delta}(t-x)^2\,p_{nk}(t)\,dt \\ &\quad\leqslant \frac{1}{\delta^2}M_n((t-x)^4,x)\leqslant\frac{c_1}{\delta^2 n^2} \end{aligned}$$

where c_1 is a constant (see [4]), hence

$$M_n(|t-x|,x)\geqslant\frac{x(1-x)}{n\,\delta}-\frac{c_1}{n^3\,\delta^3}. \tag{4.2}$$

Choose $\delta = 2\sqrt{c_1/nx(1-x)}$. We obtain from (4.1) and (4.2) that

$$\frac{3(x(1-x))^{3/2}}{8\sqrt{c_1 n}} \leqslant M_n(|t-x|, x) \leqslant \frac{(2c_1+1)(x(1-x))^{1/2}}{\sqrt{nc_1}}. \tag{4.3}$$

On the other hand, from (1.2), since $V_{x-\beta}^{x+\alpha}(f) = \alpha + \beta$, it follows that

$$\begin{aligned} |M_n(f, x) - f(x)| &\leqslant \frac{5(x(1-x))^{-1}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}(f) \\ &\leqslant \frac{5(x(1-x))^{-1}}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant \frac{10(x(1-x))^{-1}}{\sqrt{n}}. \end{aligned} \tag{4.4}$$

Hence by comparing (4.3) and (4.4), we see that (1.2) cannot be asymptotically improved for functions of bounded variation.

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